TERES - Tail Event Risk Expected Shortfall

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Motivation

Figure 1: Nezha (Wikipedia - link)

Risk Management

- Tail risk measurement
  - Expected shortfall - coherent; VaR - not coherent
  - EVT data
  - Historical estimation questionable

Example: Credit risk, Deutsche Bank: low risk levels $\{0.0002, 0.001, 0.01\}$

Coherence ES and VaR

Objectives

(i) Expected Shortfall (ES)

- Conditional expectation of a r.v.
- Expectiles, quantiles and tail heaviness

(ii) TERES

- ES estimation: mixture distribution environment, robustness
- Tail scenarios and ES range: risk level, lengthening the tail

Quantile VaR

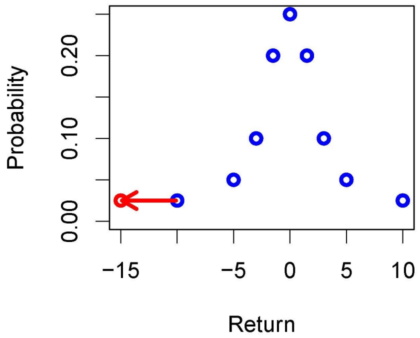

Figure 2: Distribution of returns, $\widehat{VaR}_{0.05}$ remains unchanged if tail structure changes

Expected Shortfall

Example

ES for a portfolio at 1%

(a) Standard normal, $VaR = -2.33$, $ES = -2.66$

(b) Standard Laplace, $VaR = -3.91$, $ES = -4.91$

Expected Shortfall

Example

An investor has a long position in the S&P 500 index and estimates ES at 1% level, 250 days rolling window

(a) Standard normal

(b) Standard Laplace

Research Questions

What are the thrills for ES estimation?

How does the risk level α influence the variability of ES estimates?

Which range of ES is expected under different tail scenarios?

Outline

1. Motivation ✓
2. Expected Shortfall
3. TERES
4. Empirical Results
5. Conclusions

Expected Shortfall

Risk Management

- Expected shortfall

$$ES_\eta = \mathsf{E}[Y|Y < \eta]$$

  - here $VaR_\alpha = q_\alpha = F^{-1}(\alpha)$
  - Example: $ES_\alpha = \mathsf{E}[Y|Y < q_\alpha]$

M-Quantiles

- Loss function $\rho_{\alpha,\gamma}(u) = |\alpha - \mathbf{I}\{u < 0\}|\,|u|^{\gamma}$
  - Quantile - ALD location estimate
    $q_\alpha = \arg\min_\theta \mathsf{E}\,\rho_{\alpha,1}(Y - \theta)$
  - Expectile - AND location estimate
    $e_\alpha = \arg\min_\theta \mathsf{E}\,\rho_{\alpha,2}(Y - \theta)$

Loss Function

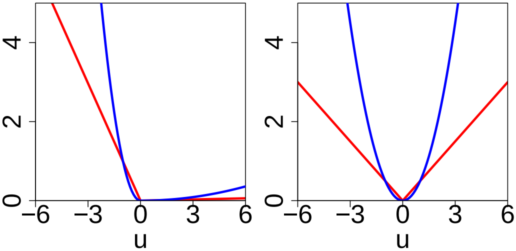

Figure 3: Expectile and quantile loss functions at $\alpha = 0.01$ (left) and $\alpha = 0.50$ (right)

LQRcheck

Tail Structure

- M-Quantiles
  - Level α, e_α and q_α
  - Level τ_α, $e_{\tau_\alpha} = q_\alpha$
- Taylor (2008) Proof

$$ES_\alpha = e_{\tau_\alpha} + \frac{e_{\tau_\alpha} - \mathsf{E}[Y]}{1 - 2\tau_\alpha} \frac{\tau_\alpha}{\alpha}$$

Expectiles and Quantiles

- Jones (1993), Guo and Härdle (2011) ▸ Proofs

$$\tau_\alpha = \frac{LPM_Y(q_\alpha) - q_\alpha \alpha}{2\{LPM_Y(q_\alpha) - q_\alpha \alpha\} + q_\alpha - \mathsf{E}[Y]}$$

$$LPM_Y(u) = \int_{-\infty}^{u} sf(s)ds$$

Example: $LPM_Y(q_\alpha) = -\varphi(q_\alpha)$ for N(0, 1)

TERES

- Flexible statistical framework - tail scenarios
- ES estimation
  - Family of distributions - environment
  - Mixture distribution for Y
  - Example: Normal-Laplace mixture

Mixture Distribution

- Contamination level $\delta \in [0,1]$, Huber (1964)

$$f_\delta(y) = (1-\delta)\varphi(y) + \delta h(y)$$

  - $h(\cdot)$ - pdf of a symmetrically distributed r.v.
  - Example: standard Laplace $h(\cdot)$

- Cases: standard normal, $\delta = 0$; standard Laplace, $\delta = 1$

Expected Shortfall

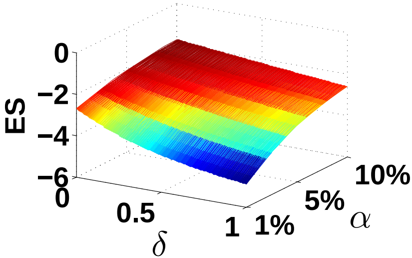

Figure 4: Theoretical ES assuming different contamination (δ) and risk levels (α)

Data

- Datastream: DAX, FTSE 100 and S&P 500
- Span: 20050103-20141231 (2609 trading days)
- Standardized daily returns - GARCH$(1,1)$

Data

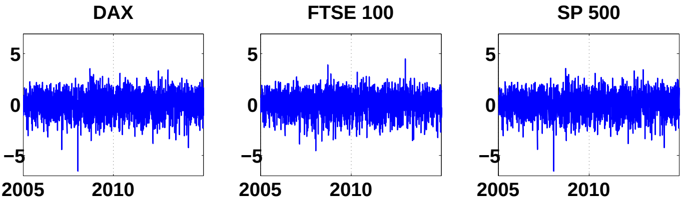

Figure 5: Standardized returns of the selected indices from 20050103-20141231

Expected Shortfall

- Risk level α: 0.01, 0.05 and 0.10
- Sample quantiles $\widehat{q}_\alpha$: -2.62, -1.43 and -1.03
- Contamination level

$$\delta \in \{0, 0.001, 0.002, 0.005, 0.01, 0.02, 0.05, 0.10, 0.15, 0.25, 0.5, 1\}$$

Expected Shortfall

δ	DAX	FTSE 100	S&P 500
0.0	-2.91	-3.11	-3.26
0.001	-2.91	-3.11	-3.26
0.002	-2.91	-3.12	-3.27
0.005	-2.92	-3.13	-3.28
0.01	-2.94	-3.14	-3.30
0.02	-2.97	-3.17	-3.33

Table 1: Estimated ES for selected indices at $\alpha = 0.01$, from 20140116-20141231 (250 trading days)

Expected Shortfall

δ	DAX	FTSE 100	S&P 500
0.05	-3.05	-3.26	-3.42
0.1	-3.16	-3.38	-3.54
0.15	-3.24	-3.46	-3.63
0.25	-3.32	-3.55	-3.72
0.5	-3.30	-3.53	-3.70
1.0	-3.19	-3.41	-3.57

Table 2: Estimated ES for selected indices at $\alpha = 0.01$, from 20140116-20141231 (250 trading days)

Expected Shortfall

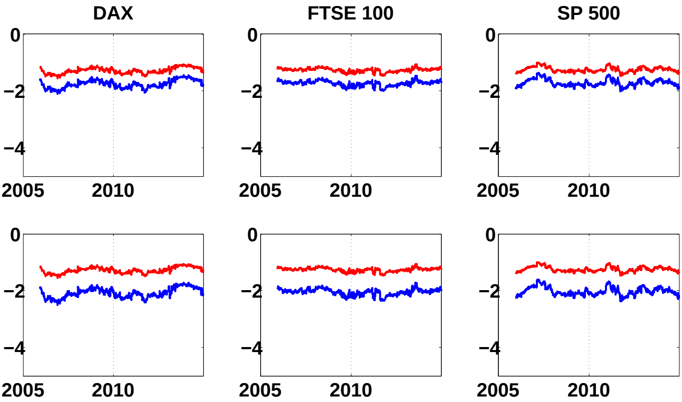

Figure 6: ES and VaR at $\alpha = 0.10$; $\delta = 0$ (top) and $\delta = 1$ (bottom); rolling window of 250 observations for the quantile

Expected Shortfall

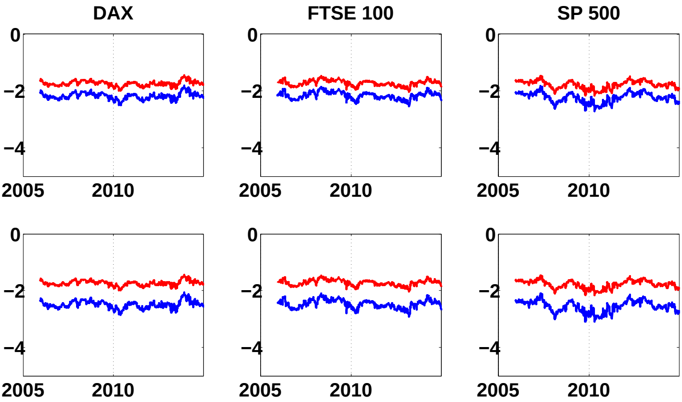

Figure 7: ES and VaR at $\alpha = 0.05$; $\delta = 0$ (top) and $\delta = 1$ (bottom); rolling window of 250 observations for the quantile

Expected Shortfall

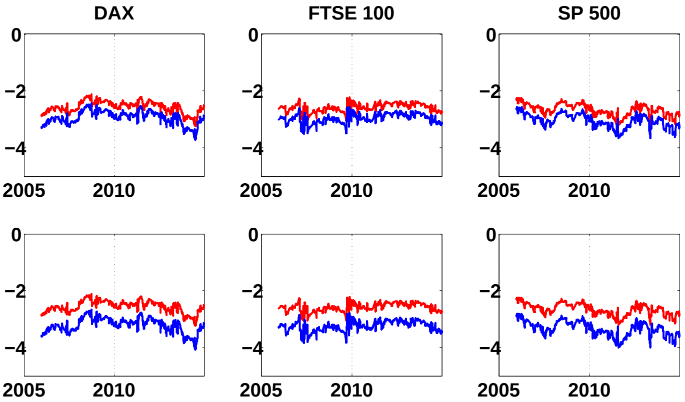

Figure 8: ES and VaR at $\alpha = 0.01$; $\delta = 0$ (top) and $\delta = 1$ (bottom); rolling window of 250 observations for the quantile

Outlook

- δ-environment
  - Strict convexity
  - Analytical formula for Normal and Laplace cases
- Connection to Generalized Error Distribution (GED)
  - Risk level α is connected to skewness
  - Integration of moments into τ estimation

GED

Conclusions

(i) Expected Shortfall (ES)

- Expectiles are successfully used for ES estimation
- ES for different α and δ illustrated

(ii) TERES

- Example: Normal-Laplace mixture distribution
- ES: DAX, FTSE 100 and S&P 500; rolling window exercise

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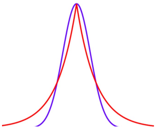

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Coherence

- Coherent risk measure $\rho(\cdot)$ of real-valued r.v.'s which model the losses
  - Subadditivity, $\rho(x+y) \leq \rho(x) + \rho(y)$ ▸ Details
  - Translation invariance, $\rho(x+c) = \rho(x)$ for a constant c
  - Monotonicity, $\rho(x) > \rho(y), \quad x < y$
  - Positive homogeneity, $\rho(kx) = k\rho(x), \quad k > 0$

▸ Risk Management

Subadditivity

▸ Coherence

- $\rho(x + y) \leq \rho(x) + \rho\{y\}$
- Diversification never increases risk
- Quantiles are not subadditive
- Expected shortfall is subadditive, Delbaen (1998)

▸ Risk Management

Expectile

$$e_{\tau_\alpha} = \arg\min_{\theta} \mathsf{E}\, \rho_{\tau_\alpha,2}\left(Y - \theta\right)$$

$$\rho_{\tau_\alpha,2}\left(u\right) = \left|\tau_\alpha - \mathbf{I}\left\{u < 0\right\}\right| \left|u\right|^2$$

This is a Quadratic convex problem with F.O.C.

$$(1 - \tau_\alpha) \int_{-\infty}^{s} (y - s) f(y) dy + \tau_\alpha \int_{s}^{\infty} (y - s) f(y) dy = 0$$

▸ Tail Structure

$$(1-\tau_\alpha)\int_{-\infty}^{e_{\tau_\alpha}}(y-e_{\tau_\alpha})f(y)dy+(1-\tau_\alpha)\int_{e_{\tau_\alpha}}^{\infty}(y-e_{\tau_\alpha})f(y)dy$$
$$=-\tau_\alpha\int_{e_{\tau_\alpha}}^{\infty}(y-e_{\tau_\alpha})f(y)dy+(1-\tau_\alpha)\int_{e_{\tau_\alpha}}^{\infty}(y-e_{\tau_\alpha})f(y)dy$$

$$(1-\tau)\{\mathsf{E}(Y)-e_{\tau_\alpha}\}=(1-2\tau_\alpha)\int_{e_{\tau_\alpha}}^{\infty}(y-e_{\tau_\alpha})f(y)dy$$
$$e_{\tau_\alpha}-\mathsf{E}(Y)=\frac{(2\tau_\alpha-1)}{1-\tau_\alpha}\int_{e_{\tau_\alpha}}^{\infty}(y-e_{\tau_\alpha})f(y)dy$$

▸ Tail Structure

$$e_{\tau_\alpha} - \mathsf{E}[Y] = \frac{1-2\tau_\alpha}{\tau_\alpha} \mathsf{E}\left[(Y - e_{\tau_\alpha})\mathbf{I}\{Y > e_{\tau_\alpha}\}\right]$$

$$\mathsf{E}[Y|Y > e_{\tau_\alpha}] = e_{\tau_\alpha} + \frac{\tau(e_{\tau_\alpha} - \mathsf{E}[Y])}{(1-2\tau_\alpha)F(e_{\tau_\alpha})}$$

And using $e_{\tau_\alpha} = q_\alpha$

$$\mathsf{E}[Y|Y > q_\alpha] = e_{\tau_\alpha} + \frac{(e_{\tau_\alpha} - \mathsf{E}[Y])\tau_\alpha}{(1-2\tau_\alpha)\alpha}$$

$$= \mathsf{ES}(e_{\tau_\alpha}, \tau_\alpha|\alpha)$$

▸ Tail Structure

Relation of Expectiles and Quantiles

$$0 = (1-\tau_\alpha)\int_{-\infty}^{e_{\tau_\alpha}} (y - e_{\tau_\alpha}) f(y) dy + \tau_\alpha \int_{e_{\tau_\alpha}}^{\infty} (y - e_{\tau_\alpha}) f(y) dy$$

Reformulation yields

$$\tau_\alpha \left(e_{\tau_\alpha} - 2\int_{-\infty}^{e_{\tau_\alpha}} e_{\tau_\alpha} f(y) dy \right) + \int_{-\infty}^{e_{\tau_\alpha}} e_{\tau_\alpha} f(y) dy$$

$$= \tau_\alpha \left(\int_{-\infty}^{\infty} y f(y) dy - 2\int_{-\infty}^{e_{\tau_\alpha}} y f(y) dy \right) + \int_{-\infty}^{e_{\tau_\alpha}} y f(y) dy$$

▸ Expectiles and Quantiles

$$\tau_\alpha \left\{ 2 \left(\int_{-\infty}^{e_{\tau_\alpha}} y f(y) dy - e_{\tau_\alpha} \int_{-\infty}^{e_{\tau_\alpha}} f(y) dy \right) + e_{\tau_\alpha} - \mathrm{E}[Y] \right\}$$
$$= \int_{-\infty}^{e_{\tau_\alpha}} y f(y) dy - \int_{-\infty}^{e_{\tau_\alpha}} e_{\tau_\alpha} f(y) dy$$

And finally

$$\tau_\alpha = \frac{\mathrm{LPM}_{e_{\tau_\alpha}}(y) - e_{\tau_\alpha} F(e_{\tau_\alpha})}{2\left\{\mathrm{LPM}_{e_{\tau_\alpha}}(y) - e_{\tau_\alpha} F(e_{\tau_\alpha})\right\} + e_{\tau_\alpha} - \mathrm{E}[Y]}$$

▸ Expectiles and Quantiles

Generalized Error Distribution

- Let $\kappa > 0$ and $g(x)$ be a symmetric distribution
- Asymmetric pdf $f(x)$

$$f(x) = \frac{2\kappa}{1+\kappa^2} \begin{cases} g(x\kappa) & , 0 \leq x \\ g(\frac{x}{\kappa}) & , \text{ else} \end{cases} \tag{1}$$

- Generalized Error Distribution (GED)

$$g(x|\gamma, \sigma, \theta) = \frac{\gamma}{2\sigma\Gamma(\frac{1}{\gamma})} \exp\left\{ - \left| \frac{x-\theta}{\sigma} \right|^{\gamma} \right\} \tag{2}$$

▸ Outlook

Ayebo and Kozubowski (2003), (1) and (2) yield a skew GED:

$$f(x|\gamma,\kappa,\sigma,\theta) = \frac{\gamma}{2\sigma\Gamma(\frac{1}{\gamma})}\frac{\kappa}{1+\kappa^2}\exp\left\{-\frac{\kappa^\gamma}{\sigma^\gamma}\left|x-\theta\right|_+^\gamma - \frac{1}{\kappa^\gamma\sigma^\gamma}\left|x-\theta\right|_-^\gamma\right\}$$

- Parameter
  - γ Shape, $\gamma = 1$ Laplace, $\gamma = 2$ Normal
  - κ Skewness, $\kappa = 1$ is symmetric
  - σ Scale
  - θ Mean

▸ Outlook

- Part of $-\log\{f(\cdot)\}$ that depends on x

$$\frac{\kappa^\gamma}{2\sigma^\gamma}|x-\theta|^\gamma \mathbf{I}\{x-\theta \le 0\} + \frac{1}{2\kappa^\gamma\sigma^\gamma}|x-\theta|^\gamma \mathbf{I}\{x-\theta < 0\}$$

- M-quantile loss function

$$\begin{aligned}\rho(x-\theta) &= |\tau - \mathbf{I}\{x-\theta<0\}||x-\theta|^\gamma \\ &= \tau|x-\theta|^\gamma \mathbf{I}\{x-\theta \le 0\} + (1-\tau)|x-\theta|^\gamma \mathbf{I}\{x-\theta<0\}\end{aligned}$$

- M-Quantile-GED relation: $\frac{\alpha}{1-\alpha} \propto \frac{\kappa^\gamma}{\kappa^{-\gamma}} = \kappa^{2\gamma}$

▸ Outlook